JRAHS Year 12 Extension 2 Term 2 Assessment 2010

QUESTION 1 **(15 Marks)** **Marks**

(a) (i) Prove that the equation of the tangent to the rectangular hyperbola $xy = 16$ at the point $A\left(4p, \frac{4}{p}\right)$ is $x + p^2 y = 8p$. **3**

(ii) If point B has co-ordinates $\left(4q, \frac{4}{q}\right)$, show that the equation of the chord AB is $x + pqy = 4(p+q)$. **2**

(iii) The tangent at A and the tangent at B intersect at T. Find the co-ordinates of T. **2**

(iv) If the chord AB passes through the point $R(8,8)$, show that the locus of T lies on the line $x + y = 4$. **2**

(b) A small object of 1 kg mass is fired vertically upwards from the ground with an initial speed of $50\ \text{ms}^{-1}$. At any instant the object is acted upon by gravity and a resistance of magnitude $\frac{1}{5}v$ where $v\ \text{ms}^{-1}$ is the speed of the object at that instant. Taking the acceleration due to gravity as $10\ \text{ms}^{-2}$, prove that:

(i) the time for the object to reach its maximum height is $5\log_e 2$ seconds. **3**

(ii) the maximum height reached above the ground is $250(1 - \log_e 2)$ metres. **3**

QUESTION 2 **(START A NEW PAGE)** **(16 Marks)** **Marks**

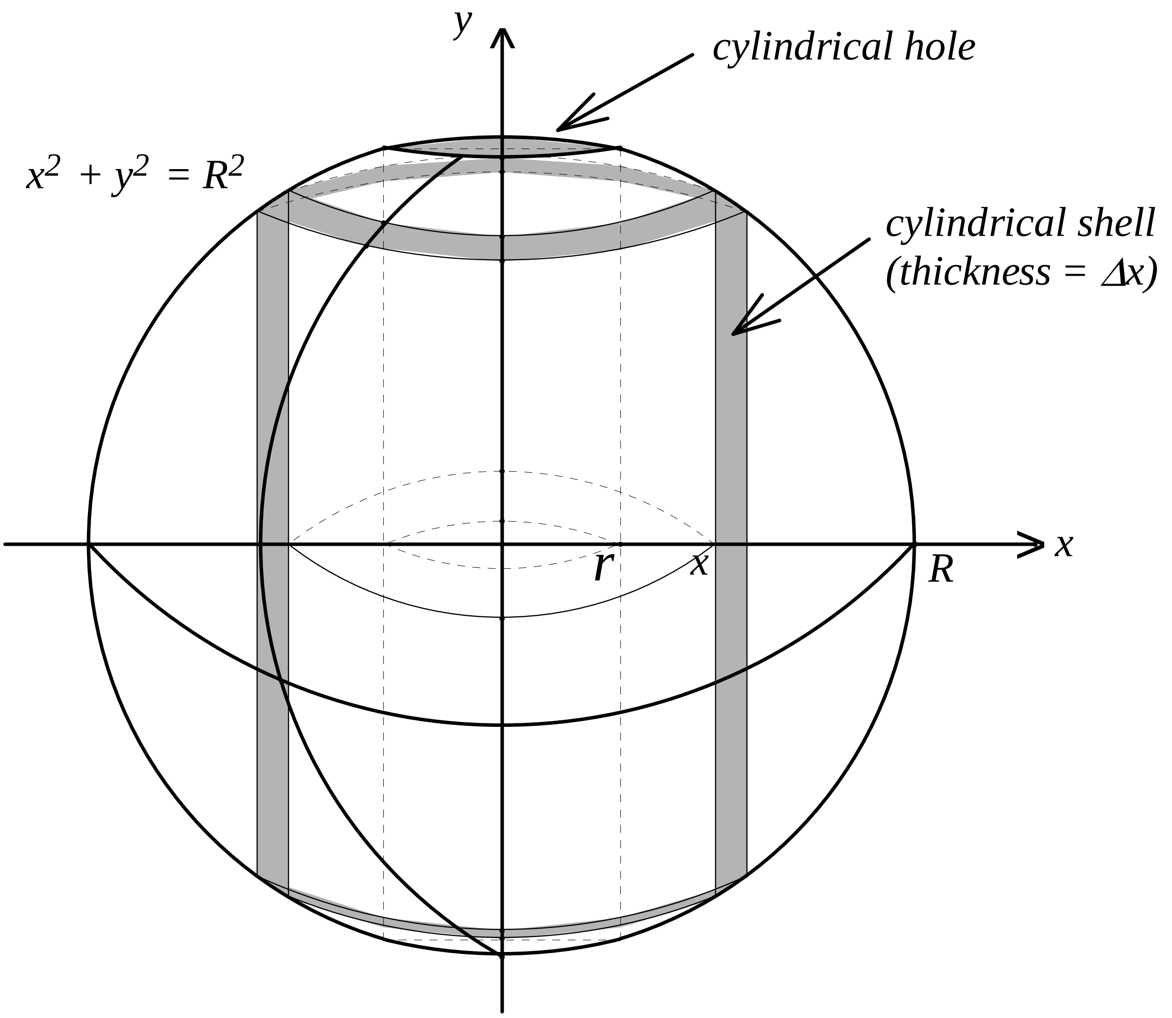
(a) (i) A right-angled isosceles triangle has hypotenuse of length h units, show that its area equals $\frac{1}{4}h^2$ square units. **1**

(ii) A solid has for its base the area bounded by the parabola $y = 0.5x^2$ and the line $y = 3x$. Cross-sections of the solid are perpendicular to the base and parallel to the $y-$axis. Each cross-section is a right-angled isosceles triangle with its hypotenuse on the base. Find the volume of the solid. **3**

QUESTION 2(b)(c) **(Continued on next page)**

QUESTION 2 **(Continued)**

(b) A solid sphere of radius R has a cylindrical hole of radius r drilled right through it such that the axis of the hole passes through the centre of the sphere.



(i) By taking cylindrical shells of width Δx and radius x, show that the volume of such a shell is approximately equal to $4\pi x\sqrt{R^2 - x^2}\ \Delta x$. **2**

(ii) Show that the volume of the remaining solid is $\frac{4}{3}\pi\left(R^2 - r^2\right)^{\frac{3}{2}}$. **3**

(c) A 1 kg mass moves in a straight line under the action of a constant driving force F Newtons. During this motion the mass encounters a resistive force of magnitude kv Newtons per unit mass where v ms^{-1} is its speed and k is a positive constant.

(i) Explain why the acceleration, $\ddot{x}$ ms^{-2}, is given by $\ddot{x} = F - kv$ for some constant $k > 0$. **1**

(ii) Given that the mass is initially at the origin with a velocity of u ms^{-1}, find a formula for its velocity, v ms^{-1}, at time t seconds. **2**

(iii) If after T seconds the velocity of the mass is $2u$ ms^{-1}, show that $F = ku\left[\dfrac{2e^{kT} - 1}{e^{kT} - 1}\right]$. **2**

(iv) Prove that the distance, d metres, moved in this time is given by $d = \dfrac{FT - u}{k}$. **2**

QUESTION 3 (START A NEW PAGE) (16 Marks) **Marks**

(a) (i) Prove that the equation the normal to the rectangular hyperbola $xy = c^2$ at the point $P\left(cp, \frac{c}{p}\right)$ is $p^2x - y = \frac{c}{p}\left(p^4 - 1\right)$. **1**

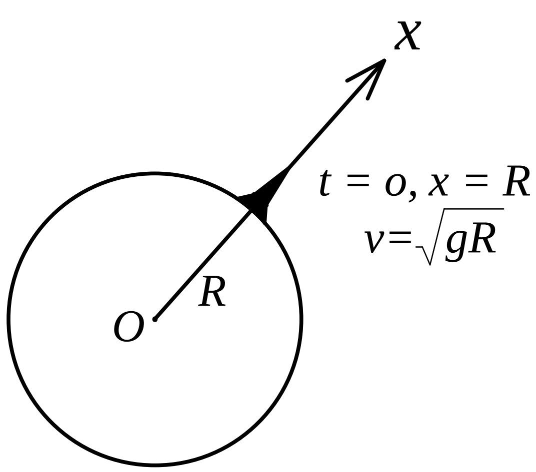
(You may assume that the equation of the tangent is $x + p^2y = 2cp$)

(ii) The tangent at $P\left(cp, \frac{c}{p}\right)$ meets the x – axis at A and the y – axis at B. Find the coordinates of A and B and show that P is the midpoint of AB. **3**

(iii) The normal at $P\left(cp, \frac{c}{p}\right)$ meets the line $y = x$ at C and the line $y = -x$ at D. Find the coordinates of points C and D. **2**

(iv) What type of quadrilateral is $ACBD$ for all $p \neq 1$? (Give reasons). **2**

(b) A small rocket is projected vertically upwards from the Earth's surface with an initial speed of $\sqrt{gR}$ ms^{-1}. You may assume that the acceleration of an object at a distance x metres from the centre of the Earth is of magnitude $\frac{gR^2}{x^2}$, and directed towards the Earth's Centre, where R metres is the radius of the Earth.



(i) Neglecting air resistance, show that the speed v ms^{-1} of the rocket at distance x metres from the Earth's centre is given by $v = \sqrt{gR}\sqrt{\frac{2R - x}{x}}$. **3**

(ii) Find the maximum height that the rocket will reach above the Earth's surface. **1**

(iii) Find the time required to reach a height of R metres above the Earth's surface. **4**

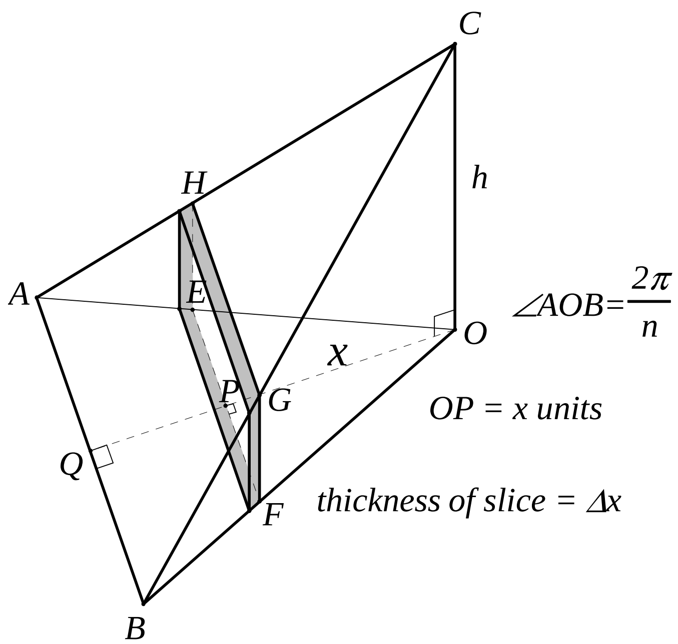
QUESTION 4 (START A NEW PAGE) (15 Marks) **Marks**

(a) A plane of mass M kg lands with velocity u ms^{-1} on a level airstrip. Upon landing it experiences a resistive force of magnitude αv^2 Newtons due to air resistance and braking, and a constant resistive force of magnitude β Newtons due to the friction between the tyres and the ground where α and β are constants.

(i) Show that the distance required to bring the plane to rest is $\dfrac{M}{2\alpha}\ln\left(1+\dfrac{\alpha}{\beta}u^2\right)$ metres. **3**

(ii) Show that the time required for the plane to come to rest is $\dfrac{M}{\sqrt{\alpha\beta}}\tan^{-1}\left(u\sqrt{\dfrac{\alpha}{\beta}}\right)$ seconds. **3**

(b) OAB is an isosceles triangle with $OA = OB = r$ and $\angle AOB = \dfrac{2\pi}{n}$. $OABC$ is a triangular pyramid with $OC = h$ and OC is perpendicular to the plane AOB.



Consider a slice with cross-section $EFGH$ perpendicular to the plane AOB with $EF \parallel AB$, thickness Δx and at a perpendicular distance x units along OQ from the point O.

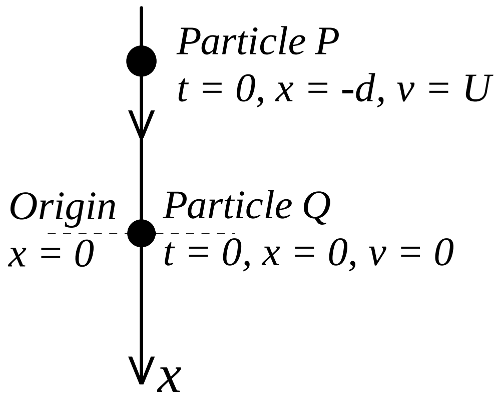
(i) Show that the volume of the slice is given by $\left(2h\tan\dfrac{\pi}{n}\right)\left(x-\dfrac{x^2}{r\cos\dfrac{\pi}{n}}\right)\Delta x$ **3**

(ii) Hence show that the volume of the pyramid is $\dfrac{hr^2}{6}\sin\dfrac{2\pi}{n}$. **4**

(iii) Suppose that n identical pyramids $OABC$ are arranged about a common vertical axis OC to form a solid. Find the limit value of the volume as n becomes very large. **2**

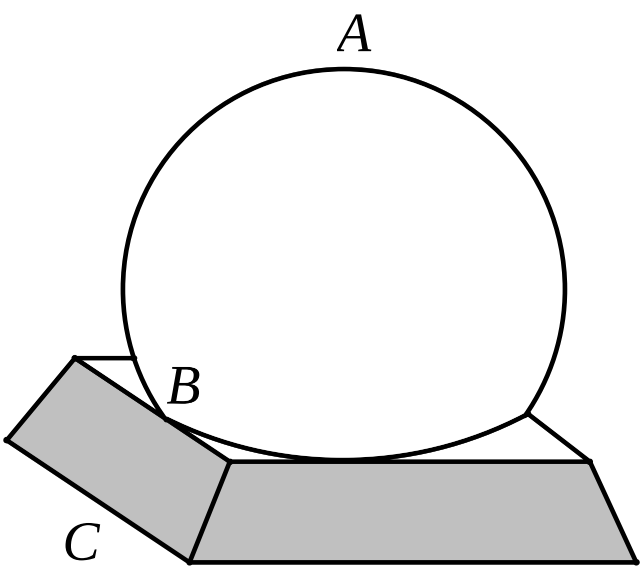
(You may assume that $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$)

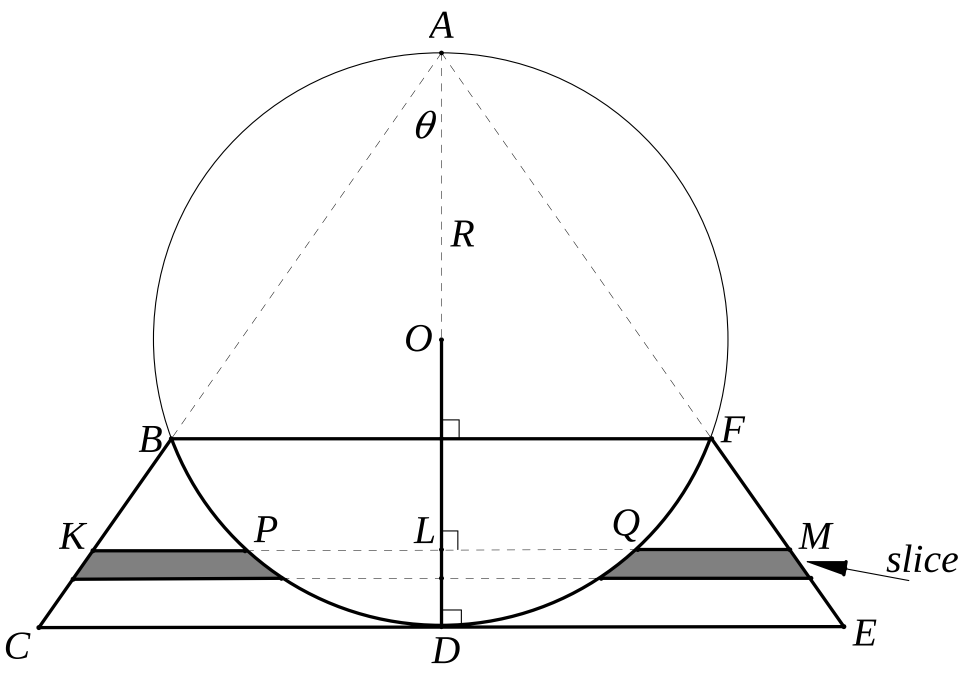
THIS IS THE END OF THE EXAMINATION

		Marks
A particle P is thrown vertically downwards in a medium where the resistive force is proportional to the speed $v\ ms^{-1}$ of the particle. The initial speed of particle P is $U\ ms^{-1}$ and the particle is thrown from a point d metres above the origin and the acceleration due to gravity is $g\ ms^{-2}$.		
(i)	Explain why the acceleration, $\ddot{x}\ ms^{-2}$, is given by $\ddot{x} = g - kv$ for some constant $k > 0$.	1
(ii)	Show that $v = \frac{g}{k} - \left(\frac{g - kU}{k}\right)e^{-kt}$.	2
(iii)	Show that $x = \frac{gt - kd}{k} - \left(\frac{g - kU}{k^2}\right)\left(e^{-kt} - 1\right)$.	2
(iv)	A second identical particle Q is dropped from the origin at the same instant that P is thrown down. Using the above results to write down similar expressions for the velocity and displacement of particle Q.	2
(v)	Find when particles P and Q collide and the speed with which they collide.	

		Marks
(i)	By considering areas or using integration techniques show that $\int_0^{\frac{a}{\sqrt{2}}} \sqrt{a^2 - x^2}\ dx = \frac{a^2(\pi + 2)}{8}$.	2
(ii)	The area of the minor segment bounded by the chord $x = \frac{a}{\sqrt{2}}$ and the circle $x^2 + y^2 = a^2$ is rotated one revolution about the chord. By considering circular cross-sections perpendicular to the chord, find the volume of the solid formed.	4

	Marks
Mr Dud’s crystal ball rests on a solid stand which is in the shape of a square based frustrum as shown.	

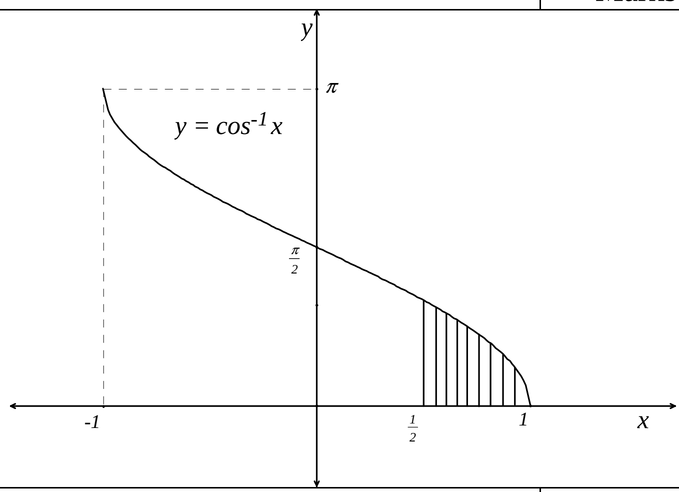


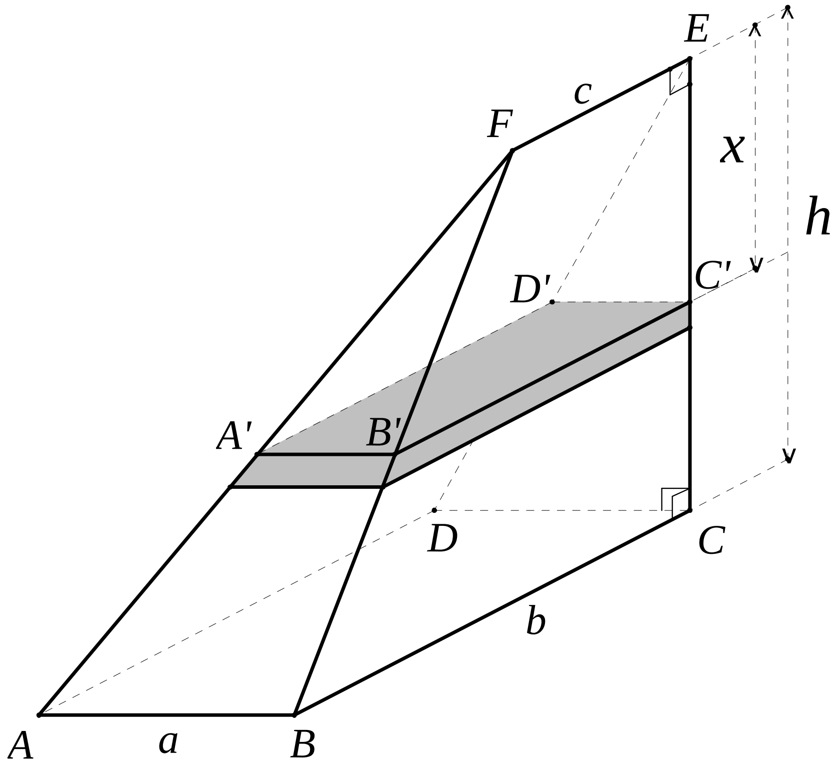


cross-section

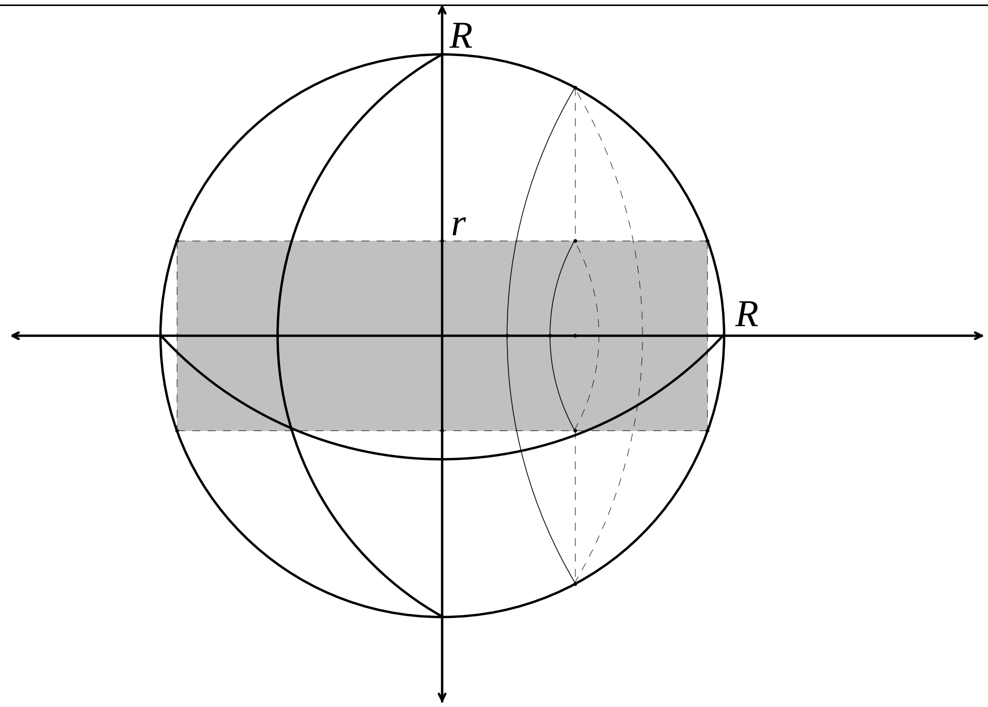
The stand is constructed so that the crystal ball of radius R fits snugly inside and just touches the centre of the square base. The side BC of the base slopes so that if extended it would pass through the top-most point of the ball at A and makes an angle θ with the vertical AD. Letting the distance OL be x units.

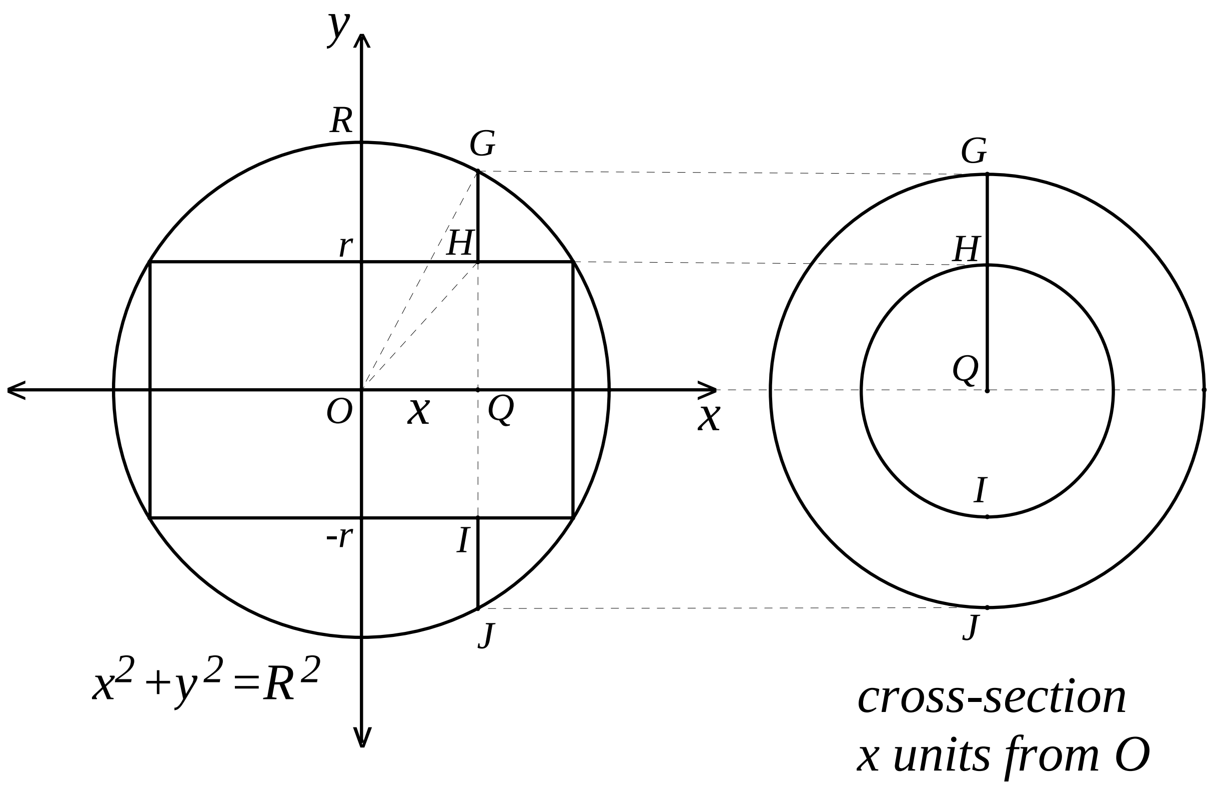
		Marks
(i)	Explain why $LQ = \sqrt{R^2 - x^2}$ and $LM = (R + x)\tan\theta$	2
(ii)	Consider a slice KLM of thickness Δx as shown perpendicular to AD. Show that it has a volume ΔV units^3 given by $\Delta V \approx \left[4\tan^2\theta(R+x)^2 - \pi(R^2 - x^2)\right]\Delta x$	2
(iii)	Find the volume of such a solid when the angle $\theta = \frac{\pi}{6}$.	3

		Marks
The region bounded by the curve $y = \cos^{-1} x$, $x = \dfrac{1}{2}$ and the $x -$ axis is rotated one revolution about the $y -$ axis to generate a solid.		
(i)	Show that an annular cross-section of the solid, parallel to the $x -$ axis with height Δy has its volume given by $\Delta V \approx \pi\left(\cos^2 y - \dfrac{1}{4}\right)\Delta y$.	2
(ii)	Hence find the volume of the solid.	3

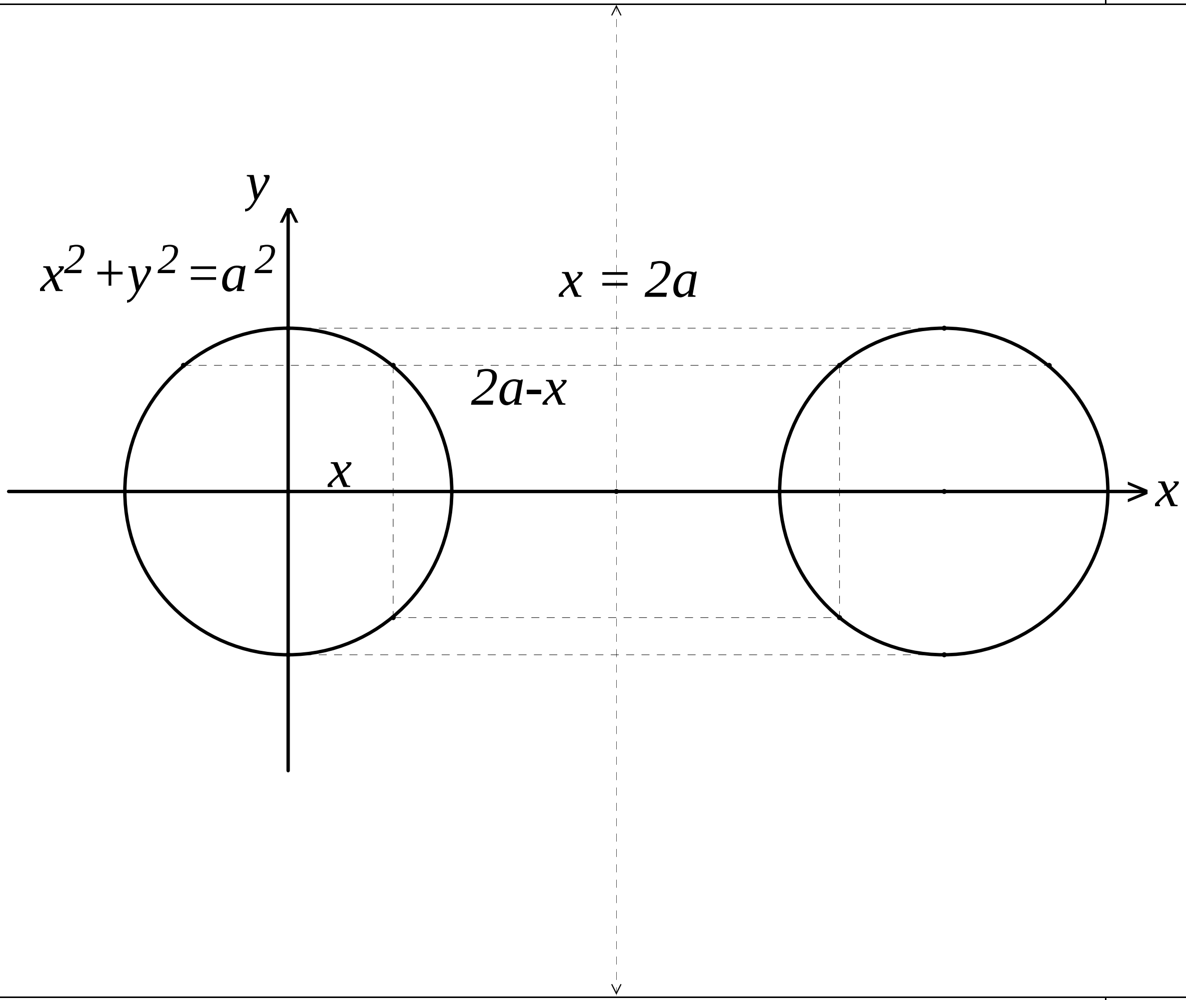
Consider solid *ABCDEF* whose height $EC = h$ and whose base is a rectangle *ABCD* with $AB = a$, $BC = b$ and top edge $EF = c$.		Marks
		
Consider a rectangular slice $A'B'C'D'$ (parallel to the base *ABCD*) *x* units from the top edge and with thickness Δx units.		
(i)	Show that the volume of the slice is given by $\Delta V = \dfrac{ax}{b}\left(c + \dfrac{b-c}{b}x\right)\Delta x$.	
(ii)	Hence prove that the volume of the solid is $\dfrac{1}{6}ha(2b+c)$.	

		Marks
A solid sphere of radius R has a cylindrical hole of radius r drilled right through it such that the axis of the hole passes through the centre of the sphere.		





(i)	By considering annular cross-sections perpendicular to the $x-axis$ and distance x units from the origin, show that the area of the annular cross-section is $\pi\left(R^2-r^2-x^2\right)$.	
(ii)	Show that the volume of the remaining solid is $\frac{4}{3}\pi\left(R^2-r^2\right)^{\frac{3}{2}}$.	

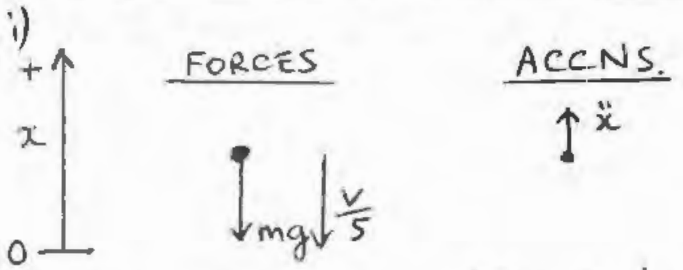
The circle $x^2 + y^2 = a^2$ is roataed one revolution about the line $x = 2a$ to form a torus.		Marks
		
(i)	By taking cylindrical shells of width Δx and axis $x = 2a$, show that the volume of such a shell is approximately equal to $4\pi(2a - x)\sqrt{a^2 - x^2}\ \Delta x$.	
(ii)	Hence show that the volume of the torus is $4\pi^2 a^3$ units^3.	

The normal at a variable point $P\left(2p, \frac{2}{p}\right)$ on the hyperbola $xy = 4$ meets the $x-axis$ at Q.		Marks
(i)	Prove that the normal at P is $p^2 x - y = \frac{2}{p}\left(p^4 - 1\right)$.	2
(ii)	Find the co-ordinates of *P*.	1
(iii)	Find the co-ordinates of *M*, the midpoint of *PQ*.	1
(iv)	Hence show that the locus of *M* lies on the curve $y^4 + xy = 2$.	2

TERM 2 2010 EXT 2 **MATHEMATICS: Question** 1 (1)

Suggested Solutions	Marks	Marker's Comments
a) i) $y = \frac{16}{x}$ $(x \neq 0)$		
$\therefore \frac{dy}{dx} = \frac{-16}{x^2}$		
$= \frac{-16}{16p^2} = -\frac{1}{p^2}$ at A	1	
$\therefore$ Tangent at A is $y - \frac{4}{p} = -\frac{1}{p^2}(x - 4p)$	1	
$p^2y - 4p = -x + 4p$		
$x + p^2y = 8p$	1	
ii) Two point form:		
$\frac{y - 4/p}{4/q - 4/p} = \frac{x - 4p}{4q - 4p}$		
$\frac{pqy - 4q}{4p - 4q} = \frac{x - 4p}{4q - 4p}$		
$pqy - 4q = -(x - 4p)$ $(p \neq q)$		
$x + pqy = 4(p+q)$	2	
iii) Tangent at B is $x + q^2y = 8q$ ①		
Solve at T $x + p^2y = 8p$ ②		
①−② $(q^2 - p^2)y = 8(q - p)$		
$(q + p)y = 8$ $(p \neq q)$		½ off if $p \neq q$ not mentioned at all.
$y = \frac{8}{p+q}$	1	
Substitute into ①		
$x = 8q - \frac{8q^2}{p+q}$		
$= \frac{8pq + 8q^2 - 8q^2}{p+q}$		½ off if T not gathered together at the end.
$= \frac{8pq}{p+q}$	1	
$\therefore$ T is $\left(\frac{8pq}{p+q}, \frac{8}{p+q}\right)$		

MATHEMATICS: Question........ (2)

Suggested Solutions	Marks	Marker's Comments
iv) The chord in (ii) goes through (8,8) $\therefore\ 8 + 8pq = 4(p+q)$ $\underline{2(1+pq) = (p+q)}$	½	
Substitute T into $x+y=4$ LHS $x+y = \frac{8pq}{p+q} + \frac{8}{p+q}$ $= \frac{8(pq+1)}{p+q}$ $= \frac{8(p+q)/2}{p+q}$ $= 4 \qquad (p \neq -q)$ $=$ RHS $\therefore$ T lies on the line $x+y=4$	1½	
b) i)  $m\ddot{x} = -mg - \frac{v}{5}$ (Newton's 2nd Law) But $m=1,\ g=10$ $\therefore\ \ddot{x} = \frac{dv}{dt} = -10 - \frac{v}{5}$	1	Diagrams were poorly done. For a given result, something was expected.
$\frac{dv}{dt} = -\frac{50+v}{5}$	1	
$\therefore \int_{50}^{0} \frac{dv}{50+v} = \int_{0}^{T} \frac{-dt}{5}$ (Where T is the required time) $\therefore \left[\ln(50+v)\right]_{50}^{0} = \left[-\frac{t}{5}\right]_{0}^{T}$		

MATHEMATICS Extension 2: Question........ (3)		
Suggested Solutions	**Marks**	**Marker's Comments**
$\therefore \ln 50 - \ln 100 = -\frac{T}{5}$		
$\ln \frac{50}{100} = -\frac{T}{5}$		
$\ln \frac{100}{50} = \frac{T}{5}$		
$\therefore T = 5 \ln 2$	1	
$\therefore$ Time to zero velocity is $5 \ln 2$ secs.		
ii) $v\frac{dv}{dx} = -10 - \frac{v}{5}$ (Different form of i)	1	
$\int_{50}^{0} \frac{v\,dv}{50+v} = -\int_{0}^{H} \frac{dx}{5}$ where H is the maximum height		
$\int_{50}^{0} \frac{(50+v)-50}{(50+v)}\,dv = -\left[\frac{x}{5}\right]_{0}^{H}$	1	
$\left[v - 50\ln(50+v)\right]_{50}^{0} = -\frac{H}{5}$		
$-50\ln 50 - 50 + 50\ln 100 = -H/5$		
$50\ln\left(\frac{100}{50}\right) - 50 = -\frac{H}{5}$		
$\therefore H = 5(50 - 50\ln 2)$		
$= 250(1 - \ln 2)$	1	
$\therefore$ Particle reaches maximum height of $250(1-\ln 2)$ metres.		

MATHEMATICS Extension 2: Question 2

Suggested Solutions	Marks	Marker's Comments
2 a (i) $\xrightarrow{F} \xleftarrow{mkv}$ $\longrightarrow +x$.		Confusion over kv & mkv.
$\Sigma F = F - mkv$	1	
$m\ddot{x} = F - mkv$		
$m=1$ $\quad \ddot{x} = F - kv$.		
(ii) $\frac{dv}{dt} = F - kv$		Difficulty students writing u & v
$\int_0^t dt = \int_u^v \frac{dv}{F-kv}$		
$t = \frac{1}{k}\ln\left(\frac{F-ku}{F-kv}\right)$	$1\frac{1}{2}$	| | need qualification $F > kv$.
$v = \frac{1}{k}\left\{F - (F - ku)e^{-kt}\right\}$	$\frac{1}{2}$	
(iii) $T = \frac{1}{k}\ln\left(\frac{F-ku}{F-2ku}\right)$		
$kT = \ln\left(\frac{F-ku}{F-2ku}\right)$		$\frac{1}{2}$ off each algebra error
$F - ku = Fe^{kT} - 2ku\,e^{kT}$	2	
$F = ku\left[\frac{2e^{kT}-1}{e^{kT}-1}\right]$		
(iv) $v\frac{dv}{dx} = F - kv$		
$\int_0^d dx = \int_u^{2u} \frac{v\,dv}{F-kv}$		
$d = \frac{-1}{k}\left[u + \frac{F}{k}\ln\left(\frac{F-2ku}{F-ku}\right)\right]$	1	
Substitute $v = 2u$ with c(ii) $T = \frac{1}{k}\ln\left[\frac{F-ku}{F-2ku}\right]$.	$\frac{1}{2}$	T requires justification
$\therefore d = \frac{-1}{k}\left[u - F.T\right]$	$\frac{1}{2}$	
distance $d = \frac{FT-u}{k}$		

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MATHEMATICS Extension 2: Question 3

Suggested Solutions	Marks	Marker's Comments
(a)		
(i) Equation of tangent $x + p^2 y = 2cp$	1	
$m_T = \frac{-1}{p^2}$		
$m_N = p^2$		
Equation of normal:		
$y - y_1 = m(x - x_1)$		
at $P(cp, \frac{c}{p})$		
$y - \frac{c}{p} = p^2(x - cp)$		
$p^2 x - y = cp^3 - \frac{c}{p}$		
$\therefore p^2 x - y = \frac{c}{p}(p^4 - 1)$		
Alternatively		
Equation of perpendicular line to tangent is		
$p^2 x - y = K$		
Sub $P(cp, \frac{c}{p})$		
$p^2(cp) - \frac{c}{p} = K$		some working must be shewn
$\therefore p^2 x - y = cp^3 - \frac{c}{p}$		
$p^2 x - y = \frac{c}{p}(p^4 - 1)$	3	
(ii) at A $y = 0$ tangent $x + p^2 y = 2cp$		
$x = 2cp$. $A(2cp, 0)$		1 A
at B $x = 0$ $p^2 y = 2cp$		1 B
$y = \frac{2c}{p}$		
$B(0, \frac{2c}{p})$		

MATHEMATICS Extension 2: Question. 3

Suggested Solutions	Marks	Marker's Comments
midpoint of AB		(1) Proof must show working to calculate midpoint AB coordinates
$x = \frac{2cp+0}{2} = cp$		
$y = \frac{0+\frac{2c}{p}}{2} = \frac{c}{p}$		
$\therefore$ P is midpoint of AB.		
(iii) Normal $p^2x - y = \frac{c}{p}(p^4-1)$	(2)	
at C $y = x$		
$p^2x - x = \frac{c}{p}(p^4-1)$		
$x(p^2-1) = \frac{c}{p}(p^2-1)(p^2+1)$		
$x = \frac{c}{p}(p^2+1)$ $p^2-1 \neq 0$		(½) for x
$\therefore y = \frac{c}{p}(p^2+1)$		(½) for y no marks lost if $p^2-1 \neq 0$ omitted
C $\left[\frac{c}{p}(p^2+1), \frac{c}{p}(p^2+1)\right]$		
at D $y = -x$		
Normal $p^2x - y = \frac{c}{p}(p^4-1)$		
$p^2x + x = \frac{c}{p}(p^4-1)$		
$x(p^2+1) = \frac{c}{p}(p^2-1)(p^2+1)$		
$x = \frac{c}{p}(p^2-1)$		
$y = -\frac{c}{p}(p^2-1) = \frac{c}{p}(1-p^2)$		(½) for x
D $\left[\frac{c}{p}(p^2-1), \frac{c}{p}(1-p^2)\right]$		(½) for y

MATHEMATICS Extension 2: Question 3

Suggested Solutions	Marks	Marker's Comments
(iv) Find midpoint of CD		
$x = \left[\frac{c}{p}(p^2+1) + \frac{c}{p}(p^2-1)\right] \div 2$		
$= \frac{c}{p}\left[\frac{2p^2}{2}\right] = cp$		
$y = \left[\frac{c}{p}(p^2+1) + \frac{c}{p}(1-p^2)\right] \div 2$		
$= \frac{c}{p}\left[\frac{2}{2}\right] = \frac{c}{p}$		
$\therefore$ P is midpoint of CD.		
$\therefore$ Diagonals bisect each other		
ACBD is a parallelogram.		
as AB is on the tangent CD is on the normal.		
$\therefore$ AB is perpendicular to CD.		
ACBD is a rhombus. (diagonals bisect at right angles)		If correct reasoning given
TEST angle between sides BC and AC		($\frac{1}{2}$) kite
$m_{BC} = \dfrac{\frac{c}{p}(p^2+1) - \frac{2c}{p}}{\frac{c}{p}(p^2+1) - 0}$		($\frac{1}{2}$) parallelogram
		(1) rhombus
		(1) rectangle
$= \dfrac{\frac{c}{p}(p^2+1-2)}{\frac{c}{p}(p^2+1)} = \dfrac{p^2-1}{p^2+1}$		(2) square
$m_{AC} = \dfrac{c/p\,(p^2+1)}{c/p\,(p^2+1) - 2cp}$		
$= \dfrac{\frac{c}{p}(p^2+1)}{c/p\,(p^2+1-2p^2)} = \dfrac{p^2+1}{1-p^2}$		
$\therefore m_{BC} \times m_{AC} = -1$ $\therefore$ BC is perpendicular to AC.		
$\therefore$ ACBD is a square. rhombus with one right angle.		

MATHEMATICS Extension 2: Question 3

Suggested Solutions	Marks	Marker's Comments
3(b) i $\dfrac{d(\frac{1}{2}v^2)}{dx} = \dfrac{-gR^2}{x^2}$		
$\displaystyle\int_{\sqrt{gR}}^{v} d(\tfrac{1}{2}v^2) = -gR^2\int_R^x \frac{1}{x^2}\,dx$		① integral
$\left[\tfrac{1}{2}v^2\right]_{\sqrt{gR}}^{v} = -gR^2\left[-\dfrac{1}{x}\right]_R^x$		½ integrating
$\tfrac{1}{2}v^2 - \tfrac{1}{2}gR = gR^2\left[\dfrac{1}{x} - \dfrac{1}{R}\right]$		½ sub limits (or finding c)
$v^2 = gR^2\left[\dfrac{2(R-x)}{Rx}\right] + gR$		
$v^2 = gR^2\left[\dfrac{2R-2x+x}{Rx}\right]$		½ simplifying
$= gR^2\left[\dfrac{2R-x}{Rx}\right]$		
$v = \sqrt{gR\left(\dfrac{2R-x}{x}\right)}$ as $v>0$ for speed.		½ $v>0$.
(ii) maximum height occurs when $v=0$		
$0 = \sqrt{gR\left(\dfrac{2R-x}{x}\right)}$		
$\therefore x = 2R$		½ $x=2R$
Max height above Earth $= 2R - R$		
$= R$ metres.		½ $H=R$

MATHEMATICS Extension 2: Question 3

Suggested Solutions	Marks	Marker's Comments
3(b)(iii) $v = \sqrt{gR}\sqrt{\frac{2R-x}{x}}$	4	
$\frac{dx}{dt} = \sqrt{gR}\sqrt{\frac{2R-x}{x}}$		1 integral
$\int_R^{2R} \frac{\sqrt{x}}{\sqrt{2R-x}}\,dx = \sqrt{gR}\int_0^T dt$		
Method 1		
$\int_R^{2R} \frac{x}{\sqrt{2Rx-x^2}}\,dx = \sqrt{gR}\,T$		
$\therefore \sqrt{gR}\,T = \frac{1}{2}\int_R^{2R}\left(\frac{2x-2R}{\sqrt{2Rx-x^2}} + \frac{2R}{\sqrt{2Rx-x^2}}\right)dx$		1 separation of fraction
$= \left[-\frac{1}{2}\times 2\sqrt{2Rx-x^2}\right]_R^{2R} + R\sin^{-1}\left[\frac{x-R}{R}\right]_R^{2R}$		1 integration
$= \left[0 + R\sin^{-1}(1)\right] - \left[-\sqrt{R^2} - 0\right]$		
$= R\left(\frac{\pi}{2}+1\right)$		1 answer
$\therefore$ Time $= \frac{\sqrt{R}}{\sqrt{g}}\left[\frac{\pi}{2}+1\right]$		

MATHEMATICS Extension 2: Question 3

Suggested Solutions	Marks	Marker's Comments
3(b)(iii) Alternative method (2)	(4)	
$\sqrt{gR}\int_0^T dt = \int_R^{2R} \frac{\sqrt{x}}{\sqrt{2R-x}}\,dx$		① integral
Let $x = 2R\sin^2\theta$ $\quad x=2R\ \ \theta=\frac{\pi}{2}$		
$\frac{dx}{d\theta} = R\sin\theta\cos\theta$ $\quad x=R\ \ \theta=\frac{\pi}{4}$		
$\sqrt{gR}\,T = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{\frac{2R\sin^2\theta}{2R-2R\sin^2\theta}} \times 4R\sin\theta\cos\theta\,d\theta$		① correct substitution and limits
$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{\frac{\sin^2\theta}{\cos^2\theta}} \times 4R\sin\theta\cos\theta\,d\theta$		
$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4R\tan\theta\sin\theta\cos\theta\,d\theta$		
$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4R\sin^2\theta\,d\theta$		
$= 4R\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1-\cos 2\theta}{2}\,d\theta$		
$= 2R\left[\theta - \frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$		① integration
$= 2R\left\{\left[\frac{\pi}{2} - 0\right] - \left[\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right]\right\}$		
$= R\left[\frac{\pi}{2} + 1\right]$		① answer.
Time $= \frac{\sqrt{R}}{\sqrt{g}}\left[\frac{\pi}{2} + 1\right]$		

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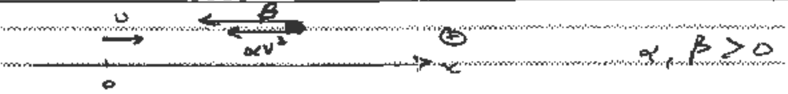
MARKING SCHEME

MATHEMATICS Extension 2: Question 4

Q4 — Suggested Solutions | Marks | Marker's Comments

(a) (i)



$\alpha, \beta > 0$

Let $t=0$ on landing, $x=0$, $v=u$

Let $t=T$, $x=D$ when $v=0$

Force EQU: $M\ddot{x} = -\beta - \alpha v^2$ (Newton's II) ✓ — **1**

$\ddot{x} = -\frac{1}{M}(\beta + \alpha v^2)$

$\frac{d}{dx}\frac{1}{2}v^2 = v\frac{dv}{dx} = -\frac{1}{M}(\beta+\alpha v^2)$

$\therefore \frac{v\,dv}{\beta+\alpha v^2} = \frac{-1}{M}dx$

so $\int_u^0 \frac{v\,dv}{\beta+\alpha v^2} = -\frac{1}{M}\int_0^D dx$

ie $\frac{1}{2\alpha}\left[\ln(\beta+\alpha v^2)\right]_u^0 = -\frac{1}{M}[x]_0^D = -\frac{D}{M}$ ✓ — **1**

$\therefore -\frac{D}{M} = \frac{1}{2\alpha}\left[\ln\beta - \ln(\beta+\alpha u^2)\right]$

$\therefore D = \frac{M}{2\alpha}\left[\ln\left(\frac{\beta+\alpha u^2}{\beta}\right)\right]$ ✓ — **1**

Distance travelled $= \frac{M}{2\alpha}\ln\left(1+\frac{\alpha}{\beta}u^2\right)$ m qed.

Marker's Comments: $\frac{1}{2\alpha}\ln(\beta+\alpha v^2) = \frac{-x}{M} + C$; $x=0$ $v=u$; $C = \frac{1}{2\alpha}\ln(\beta+\alpha u^2)$ etc.

using log laws implicitly

[3]

(ii) $\ddot{x} = \frac{dv}{dt} = -\frac{1}{M}(\beta+\alpha v^2)$

$\therefore \frac{dv}{\beta+\alpha v^2} = \frac{-1}{M}dt$

so $\int_u^0 \frac{dv}{\beta+\alpha v^2} = -\frac{1}{M}\int_0^T dt$ ✓ — **1**

$\frac{1}{\alpha}\int_u^0 \frac{dv}{\frac{\beta}{\alpha}+v^2} = -\frac{1}{M}[t]_0^T = \frac{-T}{M}$

$\frac{1}{\alpha}\cdot\frac{1}{\sqrt{\frac{\beta}{\alpha}}}\left[\tan^{-1}\frac{v}{\sqrt{\frac{\beta}{\alpha}}}\right]_u^0 = \frac{-T}{M}$ — **1**

ie $\frac{1}{\sqrt{\alpha\beta}}\tan^{-1}\left(v\sqrt{\frac{\alpha}{\beta}}\right)\Big]_u^0 = \frac{-T}{M}$

so $-\frac{T}{M} = \frac{1}{\sqrt{\alpha\beta}}\left[\tan^{-1}(0) - \tan^{-1}\left(u\sqrt{\frac{\alpha}{\beta}}\right)\right]$ ✓ — **1**

$\therefore$ time taken $= \frac{M}{\sqrt{\alpha\beta}}\tan^{-1}\left(u\sqrt{\frac{\alpha}{\beta}}\right)$ seconds — qed.

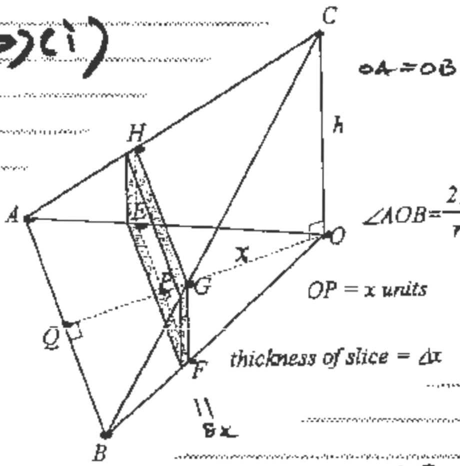
Marker's Comments: $\frac{1}{\alpha}\cdot\frac{1}{\sqrt{\frac{\beta}{\alpha}}}\tan^{-1}\frac{v}{\sqrt{\frac{\beta}{\alpha}}} = -\frac{t}{M} + C$; $t=0$ $v=u$; $\therefore C = \frac{1}{\sqrt{\alpha\beta}}\tan^{-1}u\sqrt{\frac{\alpha}{\beta}}$.

to show why $\frac{1}{\sqrt{\alpha\beta}}$ (not std $\int$)

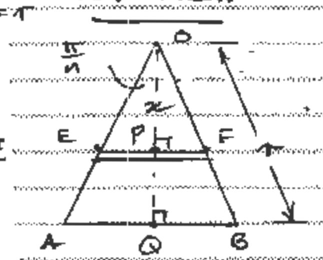
[3]

Suggested Solutions	Marks	Marker's Comments

(b)(i)



$OA = OB = r$ TOP VIEW Isos $\triangle AOB$



$\tan\frac{\pi}{n} = \frac{PF}{x}$

$PF = x\tan\frac{\pi}{n}$ — 1

$\therefore EF = 2x\tan\frac{\pi}{n}$ — $\frac{1}{2}$ — ($1\frac{1}{2}$)

or $\cos\frac{\pi}{n} = \frac{OQ}{r}$; $\sin\frac{\pi}{n} = \frac{QB}{r}$

$\therefore OQ = r\cos\frac{\pi}{n}$; $QB = r\sin\frac{\pi}{n}$ $\frac{1}{2}$

$AB = 2r\sin\frac{\pi}{n}$ $\frac{1}{2}$

$\triangle POF \ ||| \ \triangle QOB$ (equiangular)

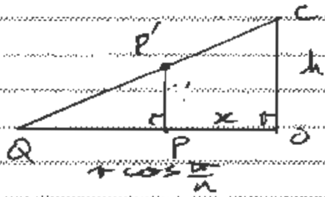
$\therefore \frac{PF}{QB} = \frac{OP}{OQ}$ (corresp. sides in similar triangles are in same ratio) $\frac{1}{2}$

$\frac{PF}{r\sin\frac{\pi}{n}} = \frac{x}{r\cos\frac{\pi}{n}}$

$PF = x\tan\frac{\pi}{n}$ $\frac{1}{2}$

$\therefore EF = 2x\tan\frac{\pi}{n}$

SIDE VIEW



$\triangle QPP' \ ||| \ \triangle QOC$ (equiangular)

$\therefore \frac{PP'}{h} = \frac{QP}{QO}$ (corresp. sides in similar triangles are in same ratio) $\frac{1}{2}$

$= \frac{QO - x}{QO} = 1 - \frac{x}{QO}$

$\therefore PP' = h\left(1 - \frac{x}{r\cos\frac{\pi}{n}}\right)$ — 1

$\therefore \delta V \doteq EF \times FG \times \delta x$

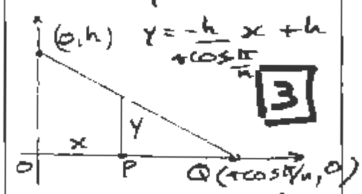
ie $\delta V \doteq 2x\tan\frac{\pi}{n} \times h\left(1 - \frac{x}{r\cos\frac{\pi}{n}}\right)\delta x$

$\delta V = 2h\tan\frac{\pi}{n}\left(x - \frac{x^2}{r\cos\frac{\pi}{n}}\right)\delta x$ qed.

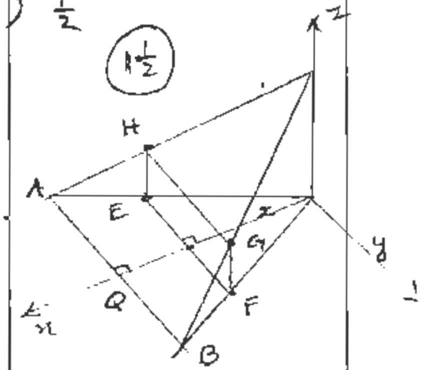
Marker's comments:

$y = mx + b$

$y = \frac{-h}{r\cos\frac{\pi}{n}}x + h$



[3] ($1\frac{1}{2}$)



(ii)

$V = \lim_{\delta x \to 0} \sum_{0}^{OQ} 2h\tan\frac{\pi}{n}\cdot\left(x - \frac{x^2}{r\cos\frac{\pi}{n}}\right)\delta x$

$OQ = r\cos\frac{\pi}{n}$ — $\frac{1}{2}$ — $OQ = r\cos\frac{\pi}{n}$

$= 2h\tan\frac{\pi}{n}\int_0^{r\cos\frac{\pi}{n}}\left(x - \frac{x^2}{r\cos\frac{\pi}{n}}\right)dx$ — [4]

$= 2h\tan\frac{\pi}{n}\left[\frac{1}{2}x^2 - \frac{x^3}{3r\cos\frac{\pi}{n}}\right]_0^{r\cos\frac{\pi}{n}}$ — 1

$= 2h\tan\frac{\pi}{n}\left[\left(\frac{r^2\cos^2\frac{\pi}{n}}{2} - \frac{r^3\cos^3\frac{\pi}{n}}{3r\cos\frac{\pi}{n}}\right) - 0\right]$ — 1

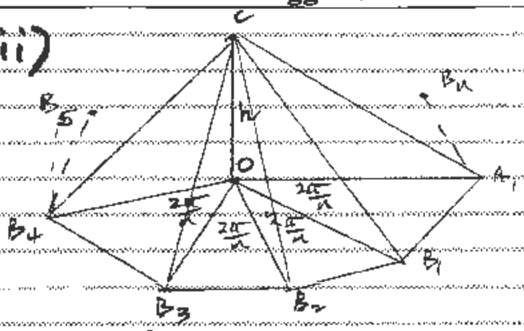
$= 2h\tan\frac{\pi}{n} \times \left[\frac{r^2\cos^2\frac{\pi}{n}}{2} - \frac{r^2\cos^2\frac{\pi}{n}}{3}\right]$

$= 2h\tan\frac{\pi}{n} \times \frac{r^2\cos^2\frac{\pi}{n}}{6} \equiv 2h\frac{\sin\frac{\pi}{n}}{\cos\frac{\pi}{n}} \times \frac{r^2\cos^2\frac{\pi}{n}}{6}$ — $\frac{1}{2}$ — show/indicate/state evidence use of double ∠s $2\sin\theta\cos\theta = \sin 2\theta$

$= \frac{r^2h}{6}\cdot 2\sin\frac{\pi}{n}\cdot\cos\frac{\pi}{n} \equiv \frac{1}{6}r^2h\sin\frac{2\pi}{n}$ qed. — 1

MATHEMATICS Extension 2: Question 4		
Suggested Solutions	**Marks**	**Marker's Comments**

(b) (iii)



Volume of n wedges / pyramids

$V_n = n \times \frac{1}{6} r^2 h \sin\frac{2\pi}{n}$ — $\frac{1}{2}$

ie $V_n = \frac{1}{6} r^2 h \, n \sin\frac{2\pi}{n}$

$\therefore V = \lim_{n\to\infty} \frac{1}{6} r^2 h \, n \sin\frac{2\pi}{n}$ — [2]

$\lim_{n\to\infty} \frac{1}{6} r^2 h . \, n \times \frac{\sin\left(\frac{2\pi}{n}\right)}{\left(\frac{2\pi}{n}\right)} \times \left(\frac{2\pi}{n}\right)$

$\lim_{n\to\infty} \frac{1}{6} r^2 h \times 2\pi \times \frac{\sin\left(\frac{2\pi}{n}\right)}{\left(\frac{2\pi}{n}\right)}$

$= \frac{1}{6} r^2 h \times 2\pi \times 1$

$V = \frac{1}{3}\pi r^2 h$

Let $x = \frac{2\pi}{n}$

$\therefore n = \frac{2\pi}{x}$

as $n \to \infty \quad x \to 0$

$\therefore \lim_{\substack{n\to\infty \\ x\to 0}} \frac{1}{6} r^2 h . \frac{2\pi}{x} . \sin x$

$\lim_{x\to 0} \frac{\pi r^2 h}{3} . \frac{\sin x}{x}$

$\therefore V = \frac{\pi}{3} r^2 h \times 1$

$V = \frac{1}{3}\pi r^2 h$ — $\frac{1}{2}$

1 For organising clearly + correctly

For small angles

as $n \to \infty \quad \frac{2\pi}{n} \to 0$

$\therefore \sin\frac{2\pi}{n} \doteqdot \frac{2\pi}{n}$ — $\frac{1}{2}$ — ? $\sin\frac{2\pi}{n} \to \frac{2\pi}{n}$

$\therefore \frac{1}{6} r^2 h n . \sin\frac{2\pi}{n} \doteqdot \frac{1}{6} r^2 h n . \frac{2\pi}{n}$ — 1

$\therefore V_n \doteqdot \frac{1}{3}\pi r^2 h \quad$ as $n \to \infty$

$\therefore V = \lim_{n\to\infty} V_n = \frac{1}{3}\pi r^2 h$